10. Nekonečná geometrická řada

Příklady

① Dána geometrická posloupnost $(a_n)_{n=1}^{\infty}$, $a_n = \left(\frac{1}{2}\right)^n$:
a) Určete prvních 6 členů, kvocient a limitu

$$a_n = \left(\frac{1}{2}\right)^n = \frac{1}{2}\cdot\left(\frac{1}{2}\right)^{n-1}$$

$$\text{GP: } a_n = \frac{1}{2}\cdot\left(\frac{1}{2}\right)^{n-1} \Rightarrow a_1 = \frac{1}{2},\ q = \frac{1}{2}$$

$$a_n = a_1 q^{n-1} \qquad \underbrace{|q|<1}_{\text{podmínka konvergence}} \Rightarrow \lim_{n\to\infty}\left(\frac{1}{2}\right)^n = 0$$

členy: $a_1 = \frac{1}{2}$ $\quad a_2 = \frac{1}{4}$ $\quad a_3 = \frac{1}{8}$ $\quad a_4 = \frac{1}{16}$ $\quad a_5 = \frac{1}{32}$ $\quad a_6 = \frac{1}{64}$

b) vytvořte novou posloupnost $(s_n)_{n=1}^{\infty}$, kde $s_n = a_1 + a_2 + \ldots + a_n$ a zjistěte, zda je konvergentní.

$$s_1 = a_1 = \frac{1}{2}$$
$$s_2 = a_1 + a_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$
$$s_3 = a_1 + a_2 + a_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$
$$s_4 = a_1 + a_2 + a_3 + a_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$$
$$\vdots$$
$$s_n = \underbrace{a_1\frac{q^n-1}{q-1}}_{\text{součet } n \text{ členů GP}} = a_1 + a_2 + \ldots + a_n = \frac{1}{2}\,\frac{\left(\frac{1}{2}\right)^n - 1}{\frac{1}{2}-1} = \frac{1}{2}\,\frac{\left(\frac{1}{2}\right)^n-1}{-\frac{1}{2}} = 1 - \left(\frac{1}{2}\right)^n$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1-\left(\frac{1}{2}\right)^n\right) = \lim_{n\to\infty} 1 - \lim_{n\to\infty}\left(\frac{1}{2}\right)^n = 1 - 0 = 1$$

VĚTA
- je-li $(a_n)_{n=1}^{\infty}$ geometrická posloupnost $[a_n = a_1 q^{n-1}]$ s kvocientem $|q|<1$, potom posloupnost částečných součtů $(s_n)_{n=1}^{\infty}$, kde $s_n = a_1 + a_2 + \ldots + a_n$, je také konvergentní a

$$\lim_{n\to\infty} s_n = \frac{a_1}{1-q}$$

- je-li $|q| \geq 1$, potom $(s_n)_{n=1}^{\infty}$ není konvergentní

[důkaz:
$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} a_1\frac{q^n-1}{q-1} = \lim_{n\to\infty}\underbrace{\frac{a_1}{q-1}}_{c\in\mathbb{R}}\cdot\lim_{n\to\infty}(q^n-1) =$$
$$= \lim_{n\to\infty}\underbrace{\frac{a_1}{q-1}}_{c\in\mathbb{R}}\Big(\underbrace{\lim_{n\to\infty} q^n}_{=0 \text{ pro } |q|<1} - \lim_{n\to\infty} 1\Big) = \frac{a_1}{q-1}(0-1) = -\frac{a_1}{q-1} = \frac{a_1}{1-q} \quad \text{cbd.}$$
]

② Dána geometrická posloupnost $((-1)^n)_{n=1}^{\infty}$. Určete několik členů posloupnosti $(s_n)_{n=1}^{\infty}$, kde $s_n = a_1 + a_2 + \ldots + a_n$ a dále zjistěte, zda je tato posloupnost konvergentní.

GP: $a_1 = -1 \quad a_2 = 1 \quad a_3 = -1 \quad a_4 = 1 \quad \ldots \quad q = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \ldots = -1$

$|q| < 1$ neplatí, podmínka konv. GP nespl. $\Big\} \Rightarrow ((-1)^n)_{n=1}^{\infty}$ je divergentní

$\Rightarrow (s_n)_{n=1}^{\infty}$ je také divergentní

$(s_n)_{n=1}^{\infty}$:
$$\left.\begin{array}{l} s_1 = a_1 = -1 \\ s_2 = a_1 + a_2 = -1+1 = 0 \\ s_3 = a_1 + a_2 + a_3 = -1+1-1 = -1 \\ s_4 = -1+1-1+1 = 0 \\ \vdots \end{array}\right\} \Rightarrow \begin{array}{l} s_{2k-1} = -1 \\ s_{2k} = 0 \end{array} \Big\} \Rightarrow$$
posloupnost částečných součtů $(s_n)_{n=1}^{\infty}$ je divergentní

③ Dána posloupnost $(a_n)_{n=1}^{\infty}$, $a_n = \frac{1}{n(n+1)}$. Určete, pokud existuje, $\lim_{n\to\infty} s_n$, kde $s_n = a_1 + a_2 + \ldots + a_n$

$a_1 = \frac{1}{2},\ a_2 = \frac{1}{6},\ a_3 = \frac{1}{12},\ a_4 = \frac{1}{20}, \ldots$ není GP $\left[\frac{a_2}{a_1} \neq \frac{a_3}{a_2} \neq q \ldots\right]$, musíme určit vzorec pro s_n

$(s_n)_{n=1}^{\infty}$:
$$\left.\begin{array}{l} s_1 = \frac{1}{2} \\ s_2 = \frac{1}{2} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3} \\ s_3 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{9}{12} = \frac{3}{4} \\ s_4 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{16}{20} = \frac{4}{5} \\ \vdots \end{array}\right\} \quad \begin{array}{l} ?\ s_n = \frac{n}{n+1} \\ ?\ \lim_{n\to\infty} s_n = 1 \end{array}$$
(dokázat matem. indukcí)

$s_n = \frac{n}{n+1}$ (předp., než jsme dok. platnost mat. ind.)

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{n \cdot 1}{n(1+\frac{1}{n})} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = \underline{1}$$

DEFINICE

Nechť je dána posloupnost $(a_n)_{n=1}^{\infty}$.
NEKONEČNOU ŘADOU nazýváme výraz $a_1 + a_2 + a_3 + \ldots + a_n + \ldots = \sum_{n=1}^{\infty} a_n$.
Členy nekonečné řady jsou členy dané posloupnosti.

- NEKONEČNÁ ŘADA je KONVERGENTNÍ $\Leftrightarrow$ posloupnost částečných součtů $(s_n)_{n=1}^{\infty}$, kde $s_n = a_1 + a_2 + \ldots + a_n$, je konvergentní a součet této řady je $s = \lim_{n\to\infty} s_n = \sum_{n=1}^{\infty} a_n$ $\quad s = \lim_{n\to\infty} s_n = \frac{a_1}{1-q}$

Př. $(2n)_{n=1}^{\infty}$

$\sum_{n=1}^{\infty} 2n = 2+4+6+8+\ldots$ divergentní

$\left(\left(-\frac{1}{3}\right)^{n-1}\right)_{n=1}^{\infty}$

$\sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^{n-1} = \overset{a_1}{1} - \overset{a_2}{\frac{1}{3}} + \overset{a_3}{\frac{1}{9}} - \frac{1}{27} + \ldots = \frac{1}{1-(-\frac{1}{3})} = \frac{1}{\frac{4}{3}} = \underline{\frac{3}{4}}$

$a_1 = 1 \quad q = \frac{a_2}{a_1} = \frac{-\frac{1}{3}}{1} = -\frac{1}{3} \quad |q| < 1 \Rightarrow$ podm. konv. spl.

DEFINICE

Je-li $(a_n)_{n=1}^{\infty}$ geometrická posloupnost s kvocientem q, $a_1 \neq 0$, pak příslušnou nekonečnou řadu

$$\sum_{n=1}^{\infty} a_n = a_1 + a_1q + a_1q^2 + a_1q^3 + \dots + a_1q^{n-1} + \dots$$

nazýváme NEKONEČNÁ GEOMETRICKÁ ŘADA

VĚTA

Je-li $(a_n)_{n=1}^{\infty}$ GEOMETRICKÁ POSLOUPNOST s kvocientem q, pak NEKONEČNÁ GEOMETRICKÁ ŘADA je pro $|q|<1$ $(a_1 \neq 0)$ KONVERGENTNÍ a její součet je $s = \lim\limits_{n\to\infty} s_n = \dfrac{a_1}{1-q} = \sum\limits_{n=1}^{\infty} a_n$

[pro $|q| \geqq 1$ je divergentní]

př. $\left(\left(-\frac{1}{3}\right)^{n-1}\right)_{n=1}^{\infty}$ [$\left(-\frac{1}{3}\right)^{1-1} = \left(-\frac{1}{3}\right)^{0} = 1 = a_1$, $a_2 = \left(-\frac{1}{3}\right)^{2-1} = -\frac{1}{3}$, $a_3 = \left(-\frac{1}{3}\right)^{2} = \frac{1}{9}$, $a_4 = -\frac{1}{27}$

=> GP: $q = -\frac{1}{3}$ $|q|<1$! podm. konv. platí =>

=> konv. i GŘ a $s = \lim\limits_{n\to\infty} s_n = \dfrac{a_1}{1-q}$]

$$s = \sum_{n=1}^{\infty}\left(-\frac{1}{3}\right)^{n-1} = 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \dots = \frac{1}{1-\left(-\frac{1}{3}\right)} = \frac{1}{\frac{4}{3}} = \frac{3}{4}$$

[$s = \lim\limits_{n\to\infty} s_n = \dfrac{a_1}{1-q}$ pro GŘ s $|q|<1$, $a_1 \neq 0$]

4) Zjistěte, zda jsou řady konvergentní, příp. určete jejich součet

a) $\sum\limits_{n=1}^{\infty} \dfrac{2n-1}{2} = \frac{1}{2} + \frac{3}{2} + \frac{5}{2} + \frac{7}{2} + \dots$ divergentní

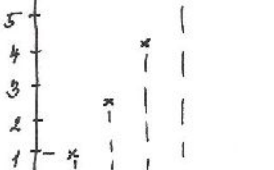
[$\lim\limits_{n\to\infty} \dfrac{2n-1}{2} = +\infty$ NEVLASTNÍ LIMITA $(+\infty, -\infty)$ => divergentní]

b) $\sum\limits_{n=1}^{\infty} \dfrac{2n-1}{n} = \frac{1}{1} + \frac{3}{2} + \frac{5}{3} + \frac{7}{4} + \frac{9}{5} + \dots$ konvergentní

$$\lim_{n\to\infty} \frac{2n-1}{n} = \lim_{n\to\infty} \frac{n\left(2-\frac{1}{n}\right)}{n} = \lim_{n\to\infty}\left(2-\frac{1}{n}\right) = 2-0 = 2$$

posloupnost $\left(\frac{2n-1}{n}\right)_{n=1}^{\infty}$ konvergentní

(navíc pro s_n ověření) => z grafu



$s_1 = 1$

$s_2 = 1 + \frac{3}{2} = \frac{5}{2} = 2{,}5$

$s_3 = 1 + \frac{3}{2} + \frac{5}{3} = \frac{6+9+10}{6} = \frac{25}{6} = 4\frac{1}{6}$

$s_4 = s_3 + a_4 = \frac{25}{6} + \frac{7}{4} = \frac{50+21}{12} = \frac{71}{12} = 5\frac{11}{12}$

$(s_n)_{n=1}^{\infty}$ divergentní (z grafu)

c) $\sum_{n=1}^{\infty} \frac{\sqrt{5}}{2^{n-1}} = \frac{\sqrt{5}}{2^0} + \frac{\sqrt{5}}{2} + \frac{\sqrt{5}}{2^2} + \frac{\sqrt{5}}{2^3} + \dots = \sqrt{5} + \sqrt{5}\cdot 2^{-1} + \sqrt{5}\cdot 2^{-2} + \sqrt{5}\cdot 2^{-3} + \dots$

(1. zp.) GŘ: $q = \frac{\frac{\sqrt{5}}{2}}{\sqrt{5}} = \frac{\frac{\sqrt{5}}{2^2}}{\frac{\sqrt{5}}{2}} = \frac{1}{2}$ podm. konv. $|q|<1$ platí

$\Rightarrow s = \lim_{n\to\infty} s_n = \frac{a_1}{1-q} = \frac{\sqrt{5}}{1-\frac{1}{2}} = \frac{\sqrt{5}}{\frac{1}{2}} = 2\sqrt{5}$

(2. zp.) upravíme

$\sum_{n=1}^{\infty} \frac{\sqrt{5}}{2^{n-1}} = \sqrt{5} + \sqrt{5}\cdot 2^{-1} + \sqrt{5}\cdot 2^{-2} + \dots$

$= \sqrt{5}\,(1 + 2^{-1} + 2^{-2} + \dots) = \sqrt{5}\,\underbrace{\left(\left(\tfrac{1}{2}\right)^0 + \left(\tfrac{1}{2}\right)^1 + \left(\tfrac{1}{2}\right)^2 + \dots\right)}_{\text{GŘ: } a_1 = 1\,[(\frac{1}{2})^0]}$

$q = \frac{1}{2}$ $|q|<1$ pl.

$\Rightarrow s = \sqrt{5}\, s' = \sqrt{5}\cdot 2 = 2\sqrt{5}$ $\qquad s' = \frac{a_1}{1-q} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$

d) $\sum_{n=1}^{\infty} \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^{n-1} = \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^0 + \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^1 + \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^2 + \dots =$

$= 1 + \frac{\sqrt{3}-1}{\sqrt{2}} + \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^2 + \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^3 + \dots$

GŘ: $a_1 = \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^0 = 1$

$q = \frac{\sqrt{3}-1}{\sqrt{2}} \doteq \frac{1{,}7-1}{1{,}4} = \frac{0{,}7}{1{,}4} \doteq \frac{1}{2}$ podm. konv. $|q|<1$ platí

$\Rightarrow s = \frac{a_1}{1-q} = \frac{1}{1-\frac{\sqrt{3}-1}{\sqrt{2}}} = \frac{1}{\frac{\sqrt{2}-\sqrt{3}+1}{\sqrt{2}}} = \frac{\sqrt{2}}{\sqrt{2}-\sqrt{3}+1}$

e) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{2}{3}\right)^n = \sum_{n=1}^{\infty} \left(-\frac{2}{3}\right)^n = \underbrace{\left(-\frac{2}{3}\right)^1 + \left(-\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^3 + \dots}$

GŘ: $q = -\frac{2}{3}$ $a_1 = -\frac{2}{3}$

$|q|<1$ platí podm. konv.

$\Rightarrow s = \frac{a_1}{1-q} = \frac{-\frac{2}{3}}{1-(-\frac{2}{3})} = \frac{-\frac{2}{3}}{\frac{5}{3}} = -\frac{6}{15} = -\frac{2}{5}$

⑤ Zjistěte, zda je nekonečná řada konvergentní a určete její součet

a) $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \dots = 1 \underbrace{+ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots}_{\text{GŘ}}$

[$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$ GP: $a_1 = 1$ $q = \frac{1}{2} \Rightarrow |q|<1$ konv.]

GŘ: $a_1 = 1$, $q = \frac{1}{2}$

$|q|<1$ podm. konv. platí $\Rightarrow$

$\Rightarrow s = \frac{a_1}{1-q} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$

b) $\sum_{n=1}^{\infty} 10^{-n} = 10^{-1}+10^{-2}+10^{-3} = \underbrace{\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}}+\ldots = \sum_{n=1}^{\infty}\left(\frac{1}{10}\right)^n$

GŘ: $q=\frac{a_2}{a_1}=\frac{\frac{1}{100}}{\frac{1}{10}}=\frac{10}{100}=\frac{1}{10}$

$a_1=\frac{1}{10}$ $\qquad |q|<1$ podmínka konvergence GŘ platí =>

$\Rightarrow s=\frac{a_1}{1-q}=\frac{\frac{1}{10}}{1-\frac{1}{10}}=\frac{\frac{1}{10}}{\frac{9}{10}}=\frac{10}{90}=\frac{1}{9}$

kdy: $s=0{,}1+0{,}01+0{,}001+\ldots=0{,}1111\ldots=0{,}\overline{1}$ $\Big\}\Rightarrow 0{,}\overline{1}=\frac{1}{9}$

⑥ napište ve tvaru zlomku s celočíselným čitatelem a jmenovatelem

a) $0{,}\overline{4}=0{,}4444\ldots=0{,}4+0{,}04+0{,}004+0{,}0004+\ldots=$

1. zp. $\quad = \underbrace{4\cdot10^{-1}+4\cdot10^{-2}+4\cdot10^{-3}+4\cdot10^{-4}+\ldots}_{GŘ}$

$q=\frac{4\cdot10^{-2}}{4\cdot10^{-1}}=10^{-1}=\frac{1}{10}$ $\qquad s=\frac{a_1}{1-q}=\frac{\frac{4}{10}}{1-\frac{1}{10}}=\frac{\frac{4}{10}}{\frac{9}{10}}=\frac{40}{90}=\frac{4}{9}$

$a_1=4\cdot10^{-1}$ $\qquad |q|<1$ platí

$0{,}\overline{4}=\frac{4}{9}$

lze i s využ. př. ⑤ b)

2. zp. $\quad 0{,}\overline{4}=0{,}4444\ldots=0{,}4+0{,}04+0{,}004+0{,}0004+\ldots=$

$= 4(\underbrace{0{,}1+0{,}01+0{,}001+0{,}0001+\ldots}_{GŘ})=$

$q=\frac{0{,}01}{0{,}1}=0{,}1=\frac{1}{10}$

$0{,}\overline{4}=4s=4\cdot\frac{1}{9}$ $\qquad a_1=0{,}1$ $\qquad s=\frac{a_1}{1-q}=\frac{\frac{1}{10}}{1-\frac{1}{10}}=\frac{\frac{1}{10}}{\frac{9}{10}}=\frac{1}{9}$

$0{,}\overline{4}=\frac{4}{9}$

[v 1. roč. jsme řešili i jinak - viz dále]

b) $-2{,}\overline{5}=-(2+0{,}5+0{,}05+0{,}005+\ldots)=-2-(\underbrace{5\cdot10^{-1}+5\cdot10^{-2}+5\cdot10^{-3}+\ldots}_{GŘ})=-2-s$

$-2{,}\overline{5}=-2-\frac{5}{9}=-\frac{23}{9}$ $\qquad q=10^{-1}$, $a_1=5\cdot10^{-1}$

$|q|<1$ pl.

$s=\frac{a_1}{1-q}=\frac{5\cdot10^{-1}}{1-\frac{1}{10}}=\frac{\frac{5}{10}}{\frac{9}{10}}=\frac{50}{90}=\frac{5}{9}$

c) $-0{,}8\overline{4}=-(0{,}8+0{,}04+0{,}004+0{,}0004+\ldots)=-(0{,}8+\underbrace{4\cdot10^{-2}+4\cdot10^{-3}+\ldots}_{GŘ})=-(0{,}8+s)$

$q=10^{-1}=\frac{1}{10}$ $\qquad a_1=4\cdot10^{-2}=\frac{4}{100}$

$-0{,}8\overline{4}=-\left(\frac{8}{10}+\frac{4}{90}\right)=-\frac{72+4}{90}=-\frac{76}{90}$ $\qquad |q|<1$ pl.

$-0{,}8\overline{4}=-\frac{38}{45}$ $\qquad s=\frac{a_1}{1-q}=\frac{\frac{4}{100}}{1-\frac{1}{10}}=\frac{\frac{4}{100}}{\frac{9}{10}}=\frac{40}{900}=\frac{4}{90}$

d) $0,\overline{12} = 0{,}121212\ldots = 12\cdot10^{-2} + 12\cdot10^{-4} + 12\cdot10^{-6} + \ldots = 12(\underbrace{10^{-2}+10^{-4}+10^{-6}+\ldots}) =$

$= 12\cdot\frac{1}{99} = \frac{12}{99} = \frac{4}{33}$

$\left[\begin{array}{l} \text{GŘ} \quad q = 10^{-2} = \frac{1}{100} \\ |q|<1 \\ \text{podm. konvergence platí} \\ s = \frac{a_1}{1-q} = \frac{10^{-2}}{1-10^{-2}} = \frac{\frac{1}{100}}{1-\frac{1}{100}} = \frac{\frac{1}{100}}{\frac{99}{100}} = \frac{1}{99} \end{array}\right]$

[nebo nevytýkat 12
$\Rightarrow a_1 = 12\cdot10^{-2} \Rightarrow s = \frac{12\cdot10^{-2}}{1-\frac{1}{100}} = \frac{12}{99}$]

e) $5{,}4\overline{87} = 5 + \frac{4}{10} + 87\cdot10^{-3} + 87\cdot10^{-5} + \ldots =$

$= \frac{54}{10} + 87(\underbrace{10^{-3}+10^{-5}+\ldots}) = \frac{54}{10} + \frac{87\cdot1}{990} = \frac{54\cdot99+87}{990} = \frac{5433}{990} = \frac{1811}{330}$

$\left[\begin{array}{l} \text{GŘ: } q = 10^{-2} \quad |q|<1 \quad \text{podm. konv. platí} \\ s = \frac{a_1}{1-q} = \frac{10^{-3}}{1-10^{-2}} = \frac{\frac{1}{1000}}{1-\frac{1}{100}} = \frac{\frac{1}{1000}}{\frac{99}{100}} = \frac{100}{99\cdot1000} = \frac{1}{990} \end{array}\right]$

nebo
$5{,}4\overline{87} = \frac{54}{10} + \underbrace{87\cdot10^{-3} + 87\cdot10^{-5} + 87\cdot10^{-7} + \ldots} = \frac{54}{10} + \frac{29}{33} = \frac{54\cdot33+290}{330} = \frac{1811}{330}$

$\left[\begin{array}{l} \text{GŘ: } q = 10^{-2} \quad |q|<1 \quad \text{podm. konv. platí} \\ s = \frac{a_1}{1-q} = \frac{87\cdot10^{-3}}{1-10^{-2}} = \frac{\frac{87}{1000}}{1-\frac{1}{100}} = \frac{\frac{87}{1000}}{\frac{99}{100}} = \frac{87\cdot100}{99\cdot1000} = \frac{87}{99} = \frac{29}{33} \end{array}\right]$

f) $-0{,}3\overline{25} = -(0{,}3 + 0{,}025 + 0{,}00025 + 0{,}0000025 + \ldots)$
3Ř/113

$= -\left(\frac{3}{10} + \frac{25}{1000} + 25\cdot10^{-5} + 25\cdot10^{-7} + \ldots\right)$

$= -\left[\frac{3}{10} + 25(\underbrace{10^{-3}+10^{-5}+10^{-7}+\ldots})\right] = -\left(\frac{3}{10} + 25\cdot\frac{1}{990}\right) =$

$\left[\begin{array}{l} \text{GŘ: } q = 10^{-2} \quad |q|<1 \quad \text{podm. konv. platí} \\ s = \frac{a_1}{1-q} = \frac{10^{-3}}{1-10^{-2}} = \frac{\frac{1}{1000}}{1-\frac{1}{100}} = \frac{\frac{1}{1000}}{\frac{99}{100}} = \frac{100}{99\cdot1000} = \frac{1}{990} \end{array}\right]$

$= -\frac{3\cdot99+25}{990} = -\frac{297+25}{990} = -\frac{322}{990} = -\frac{161}{495}$

[v 1. ROČ. ČTYŘ. (5. ROČ. OSMIL)

$0{,}3\overline{25} = 0{,}32525\ldots = \frac{3}{10} + 0{,}02525\ldots$

$a = 0{,}025|25\ldots$
$1000a = 25{,}2525\ldots$
$10a = 0{,}2525\ldots$ $\}\ominus$
$990a = 25$
$a = \frac{25}{990}$]

(7) Zjistit, pro která $x\in\mathbb{R}$ jsou řady konvergentní a určit jejich součet

3.28 a) $\sum_{n=1}^{\infty}\left(\frac{1}{x}\right)^n = \underbrace{\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \ldots} = x^{-1} + x^{-2} + x^{-3} + \ldots \qquad x \neq 0$

GŘ: $q = \frac{\frac{1}{x^2}}{\frac{1}{x}} = \frac{x}{x^2} = \frac{1}{x}$ podm. konverg. $|q|<1$

$\left|\frac{1}{x}\right| < 1$

$\frac{1}{|x|} < 1 \quad /\cdot|x|,\ |x|>0$

$1 < |x|$

$|x| > 1$, $x\neq0$

$x \in (-\infty,-1) \cup (1,+\infty)$

podmínka konv. pro x

$s = \frac{a_1}{1-q} = \frac{\frac{1}{x}}{1-\frac{1}{x}} = \frac{\frac{1}{x}}{\frac{x-1}{x}} = \frac{x}{x(x-1)} = \frac{1}{x-1}$

b) $\sum_{n=1}^{\infty} (1-5x)^n = (1-5x)^1 + (1-5x)^2 + (1-5x)^3 + (1-5x)^4 + \dots$

GŘ: $q = \frac{(1-5x)^2}{(1-5x)} = 1-5x$ podm. konverg. $|q|<1$

$|1-5x|<1$

$|5x-1|<1 \quad |:5$!

$|x-\frac{1}{5}|<\frac{1}{5}$

$x_0=\frac{1}{5}$

$0 \quad \frac{1}{5} \quad \frac{2}{5}$

$s = \frac{a_1}{1-q} = \frac{1-5x}{1-(1-5x)} = \frac{1-5x}{5x} \quad \Leftarrow \quad x \in (0, \frac{2}{5})$

⑧ Řešte v R

3.29 a) $\sum_{n=1}^{\infty} (3x)^{n-1} = 10$

$(3x)^0 + (3x)^1 + (3x)^2 + \dots = 10$

$\underbrace{1 + 3x + 9x^2 + \dots}$ $= 10$

GŘ: $q = \frac{9x^2}{3x} = \frac{3x}{1} = 3x$ podm. konv. $|q|<1$

$|3x|<1$

$3|x|<1$

$|x|<\frac{1}{3}$

$-\frac{1}{3} \quad 0 \quad \frac{1}{3}$

$s = \frac{a_1}{1-q} = \frac{1}{1-3x} \quad \Leftarrow \quad x \in (-\frac{1}{3}, \frac{1}{3})$

rovnice má tvar

$\mathcal{D} = (-\frac{1}{3}, \frac{1}{3})$

$\frac{1}{1-3x} = 10$

$1 = 10(1-3x)$

$1 = 10 - 30x$

$30x = 9$

$x = \frac{9}{30} = \frac{3}{10} \in \mathcal{D}$

$\mathcal{K} = \{\frac{3}{10}\}$

b) $\sum_{n=1}^{\infty} \left(\frac{2}{x}\right)^{n-1} = \frac{4x-3}{3x-4}$ $\quad \mathcal{O} = R, \ \mathcal{D}' = R - \{0, \frac{4}{3}\} \quad [3x-4 \neq 0, \ x \neq \frac{4}{3}]$

$\underbrace{1 + \frac{2}{x} + \left(\frac{2}{x}\right)^2 + \left(\frac{2}{x}\right)^3 + \dots}$ $= \frac{4x-3}{3x-4}$ $\quad \left[\left(\frac{2}{x}\right)^{1-1} = \left(\frac{2}{x}\right)^0 = 1 \text{ pro } x \neq 0\right]$

GŘ: $q = \frac{2}{x}$, $a_1 = 1 \rightarrow$ podm. konv. $|q|<1$ pl.

$|\frac{2}{x}|<1$

$s = \frac{a_1}{1-q} = \frac{1}{1-\frac{2}{x}} = \frac{1}{\frac{x-2}{x}}$

$\frac{2}{|x|}<1 \quad |\cdot|x|_{>0}$

$s = \frac{x}{x-2}$

$2<|x|$

$|x|>2$

$-2 \quad 0 \quad 2$

-ce má tvar: $\frac{x}{x-2} = \frac{4x-3}{3x-4}$

$x \in (-\infty, -2) \cup (2, +\infty) \cap \mathcal{D}'$

$\mathcal{D} = (-\infty, -2) \cup (2, +\infty)$

$x(3x-4) = (4x-3)(x-2)$

$3x^2 - 4x = 4x^2 - 8x - 3x + 6$

$0 = x^2 - 7x + 6$

$0 = (x-6)(x-1)$

$x_1 = 6 \in \mathcal{D} \quad x_2 = 1 \notin \mathcal{D}$

$\mathcal{K} = \{6\}$

c) $\sum_{n=1}^{\infty} (x+2)^{2n} = \frac{1}{3}$

$\underbrace{(x+2)^2 + (x+2)^4 + (x+2)^6 + \ldots} = \frac{1}{3}$

GŘ: $a_1 = (x+2)^2 \quad q = \frac{(x+2)^4}{(x+2)^2} = (x+2)^2$ podm. konv. $|q| < 1$

$|\underbrace{(x+2)^2}_{>0}| < 1$

$(x+2)^2 < 1$

$x^2 + 4x + 4 < 1$ ANULOFAT

$x^2 + 4x + 3 < 0$

$(x+3)(x+1) < 0$

$x_0 = -3 \quad x_0 = -1$

$x \in (-3, -1)$

$s = \frac{a_1}{1-q}$

$s = \frac{(x+2)^2}{1-(x+2)^2} = \frac{(x+2)^2}{[1-(x+2)][1+(x+2)]}$

$s = \frac{(x+2)^2}{(-1-x)(3+x)} = \frac{(x+2)^2}{-(1+x)(x+3)}$

$D = (-3, -1)$

rce má tvar:

$\frac{(x+2)^2}{-(1+x)(x+3)} = \frac{1}{3}$

$3(x+2)^2 = -(x+1)(x+3)$

$3(x^2 + 4x + 4) = -(x^2 + 3x + x + 3)$

$3x^2 + 12x + 12 = -x^2 - 4x - 3$

$4x^2 + 16x + 15 = 0$

$(a = 4,\ b = 16,\ c = 15)$

$x_{1,2} = \frac{-16 \pm \sqrt{16^2 - 4.4.15}}{8}$

$x_{1,2} = \frac{-16 \pm \sqrt{256 - 240}}{8} = \frac{-16 \pm 4}{8}$

$x_{1,2} = \begin{cases} -\frac{12}{8} = -\frac{3}{2} = -1{,}5 \in D \\ -\frac{20}{8} = -\frac{5}{2} = -2{,}5 \in D \end{cases}$

$K = \left\{-\frac{3}{2}, -\frac{5}{2}\right\}$

d) $\sum_{n=1}^{\infty} \frac{1}{x^{2n}} = \frac{1}{x^2 - 1}$

$\underbrace{\frac{1}{x^2} + \frac{1}{x^4} + \frac{1}{x^6} + \ldots} = \frac{1}{x^2-1}$

GŘ: $a_1 = \frac{1}{x^2}, \quad q = \frac{1}{x^2}$ podm. konvergence $|q| < 1$

$\left|\frac{1}{x^2}\right| < 1$

$\frac{1}{|x^2|} < 1$

$\frac{1}{x^2} < 1 \quad / \cdot \underset{>0}{x^2}$

$\left[\underbrace{x^2 \geq 0 \text{ pro } \forall x \in R \wedge x \neq 0}_{x^2 > 0} \Rightarrow |x^2| = x^2\right]$

$1 < x^2$

$|x| > 1$

$s = \frac{a_1}{1-q} = \frac{\frac{1}{x^2}}{1 - \frac{1}{x^2}} = \frac{\frac{1}{x^2}}{\frac{x^2-1}{x^2}} = \frac{x^2}{x^2(x^2-1)}$

$s = \frac{1}{x^2 - 1}$

$D = (-\infty, -1) \cup (1, +\infty)$

- rce má tvar: $s = \frac{1}{x^2-1}$

$\frac{1}{x^2-1} = \frac{1}{x^2-1}$

$0x = 0$

$\Rightarrow K = D = (-\infty, -1) \cup (1, +\infty)$

⑨ Řešte v R

a) $\sum_{n=1}^{\infty} (2^x)^n = 1$ $\quad [(2^x)^1 = 2^x \quad (2^x)^2 = 2^x \cdot 2^x = 2^{2x} \quad \dots]$ $\quad D' = R$

$\underbrace{2^x + 2^{2x} + 2^{3x} + \dots} = 1$

[GŘ: $a_1 = 2^x \qquad q = \frac{2^{2x}}{2^x} = \frac{2^x \cdot 2^x}{2^x} = 2^{2x-x} = 2^x$

$s = \frac{a_1}{1-q} = \frac{2^x}{1-2^x}$]

podm. konvergence $|q| < 1$

$|2^x| < 1$

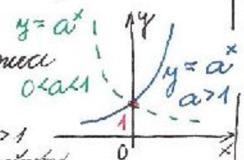
$2^x < 1$ exp. nerce

$2^x < 2^0$

$x < 0$ $(a = 2 > 1$ ⇒ neměníme zn. v nerci)

$D'' = R^-$

z grafu $|\underset{>0}{2^x}| = 2^x$



rce má tvar: $\frac{2^x}{1-2^x} = 1$

$2^x = 1 - 2^x$

$2^x + 2^x = 1$

$2 \cdot 2^x = 1$

$2^{x+1} = 1$

$2^{x+1} = 2^0$

$x + 1 = 0$

$x = -1 \in D$

$D = D' \cap D'' = R^- = (-\infty, 0)$

$K = \{-1\}$

b) $\sum_{n=1}^{\infty} \log \sqrt[2^{n-1}]{x} = 2$ $\qquad x \geq 0 \quad D' = R_0^+ - \{0\} \quad x > 0$ ($\log x$)

[$a_1 = \log \sqrt[2^0]{x} = \log \sqrt[1]{x} = x^{\frac{1}{1}} = x \qquad a_2 = \log \sqrt[2^1]{x} = \log \sqrt{x} = x^{\frac{1}{2}}$

$a_3 = \log \sqrt[2^2]{x} = \log \sqrt[4]{x} = x^{\frac{1}{4}} \qquad a_4 = \log \sqrt[2^3]{x} = \sqrt[8]{x} = x^{\frac{1}{8}} \dots$]

$\log x^1 + \log x^{\frac{1}{2}} + \log x^{\frac{1}{4}} + \log x^{\frac{1}{8}} + \dots = 2$

$1 \log x + \frac{1}{2} \log x + \frac{1}{4} \log x + \frac{1}{8} \log x + \dots = 2$

$\log x \, (\underbrace{1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots}) = 2$

[nebo (*) $\log x + \frac{1}{2}\log x + \frac{1}{4}\log x + \dots = 2$ viz dole]

[GŘ: $a_1 = 1, \; q = \frac{1}{2}$ podm. konv. $|q| < 1$ pl. vždy $x \in R$ $D'' = R$

$s = \frac{a_1}{1-q} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$]

$D = D' \cap D'' = D'$

$D = (0, +\infty)$

$2 \cdot \log x = 2$

$\log x = 1$ $\quad [\log_{10} \boxed{10} = 1 \quad 10^1 = \square$

$\log x = \log 10$

$x = 10 \in D$ $\qquad K = \{10\}$

(*) $\underbrace{\log x + \frac{1}{2}\log x + \frac{1}{4}\log x} + \frac{1}{8}\log x \dots = 2$ $\qquad D' = R' = R^+$

[GŘ: $a_1 = \log x \qquad q = \frac{\frac{1}{2}\log x}{\log x} = \frac{1}{2} \qquad |q| < 1$ pl. pro $\forall x \in R^+$

$s = \frac{a_1}{1-q} = \frac{\log x}{1-\frac{1}{2}} = \frac{\log x}{\frac{1}{2}} = 2\log x$]

rce má tvar

$2 \log x = 2$

$\vdots$

$x = 10$

(10) Řešte v R

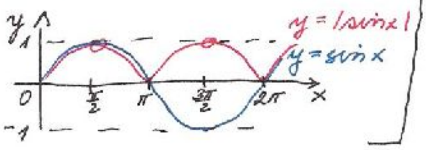
a) $\sum_{n=1}^{\infty} \sin^n x = -\frac{1}{3}$

$\underbrace{\sin x + \sin^2 x + \sin^3 x}+\dots = -\frac{1}{3}$

GŘ: $a_1 = \sin x$ $\quad q = \frac{\sin^2 x}{\sin x} = \sin x$ $\quad |q|<1$ podm. konv.

$s = \frac{a_1}{1-q} = \frac{\sin x}{1-\sin x}$ $\quad |\sin x| < 1$

$\Leftrightarrow x \neq \frac{\pi}{2} + k\pi$



$D = R - \bigcup_{k\in Z}\{\frac{\pi}{2}+k\pi\}$

na mal krae: $s = -\frac{1}{3}$

$\frac{\sin x}{1-\sin x} = -\frac{1}{3} \quad | \cdot 3(1-\sin x)$

$3\sin x = -1(1-\sin x)$

$3\sin x = -1 + \sin x$

$2\sin x = -1$

$\sin x = -\frac{1}{2}$

$\sin x = \ominus \frac{1}{2}$

$\sin x' = \frac{1}{2}$ $\quad x'$ z 1. kv.

$x' = 30° = \frac{\pi}{6}$

$\ominus$ III. kv.

$x_1 = \pi + x' + 2k\pi$

$x_1 = \pi + \frac{\pi}{6} + 2k\pi$

$x_1 = \frac{7\pi}{6} + 2k\pi \in D$

IV. kv.

$x_2 = 2\pi - x' + 2k\pi$

$x_2 = 2\pi - \frac{\pi}{6} + 2k\pi$

$x_2 = \frac{11\pi}{6} + 2k\pi \in D \quad k\in Z$

$K = \bigcup_{k\in Z}\{\frac{7\pi}{6}+2k\pi, \frac{11\pi}{6}+2k\pi\}$

b) $\sum_{n=1}^{\infty} \sin^{(2n-2)} x = 2\,tg\,x$

$a_1 = \sin^{2\cdot1-2} x = \sin^0 x = 1$

$a_2 = \sin^{2\cdot2-2} x = \sin^2 x$

$a_3 = \sin^{3\cdot2-2} = \sin^4 x$

$\underbrace{\sin^0 x}_{1} + \underbrace{\sin^2 x + \sin^4 x + \dots} = 2\,tg\,x$

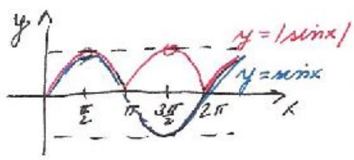
GŘ: $a_1 = 1$ $\quad q = \sin^2 x$ $\quad$ podm. konv. $|q| < 1$

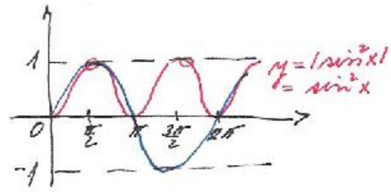
$|\sin^2 x| < 1$

$|\sin x| < 1$

pl. pro $\forall x \in R - \bigcup_{k\in Z}\{\frac{\pi}{2}+k\pi\}$

$s = \frac{a_1}{1-q} = \frac{1}{1-\sin^2 x}$





na mal drav: $s = 2\,tg\,x$

$tg\,x = \frac{\sin x}{\cos x}$

$D_{tg} = R - \bigcup_{k\in Z}\{\frac{\pi}{2}+k\pi\}$

$\frac{1}{1-\sin^2 x} = 2\,tg\,x$

$\frac{1}{\underbrace{1-\sin^2 x}_{\cos^2 x}} = 2\frac{\sin x}{\cos x}$

$\sin^2 x + \cos^2 x = 1$

$\cos^2 x = 1 - \sin^2 x$

$\sin^2 x = 1 - \cos^2 x$

$\frac{1}{\cos^2 x} = 2\frac{\sin x}{\cos x}$

$1 = 2\sin x \cos x$

$\sin 2x = 2\sin x \cdot \cos x$

$1 = \sin 2x$

$2x = \frac{\pi}{2} + 2k\pi$

$x = \frac{\pi}{4} + k\pi \in D$

$K = \bigcup_{k\in Z}\{\frac{\pi}{4}+k\pi\}$

nebo

$\sin 2x = 1$

subst. $y = 2x$

$\sin y = 1$

$y = \frac{\pi}{2} + 2k\pi$

zpět k subst.

$2x = \frac{\pi}{2} + 2k\pi$

$x = \frac{\pi}{4} + k\pi$

(11) Řešte v R

3.41 DÚ $\sum_{n=1}^{\infty} (y-1)^n = 1$ $\qquad D' = R$

$(y-1)^1 + (y-1)^2 + (y-1)^3 + \ldots = 1$

$\left[\begin{array}{l} GŘ:\ a_1 = y-1 \quad q = \frac{(y-1)^2}{y-1} = y-1 \quad \text{podm. konv. } |q|<1 \\ \qquad |y-1|<1 \\ s = \frac{a_1}{1-q} = \frac{y-1}{1-(y-1)} = \frac{y-1}{2-y} \qquad y \neq 1 \end{array} \right]$

$D'' = (0,2)$

$D = D' \cap D'' = (0,2)$

- rce má tvar: $s = 1$

$\frac{y-1}{2-y} = 1$

$y - 1 = 2 - y$

$2y = 3$

$y = \frac{3}{2} \in D$ $\qquad K = \{\frac{3}{2}\}$

(12) Vypočítejte hodnotu

a) $y = x \cdot \sqrt{x^3} \cdot \sqrt[4]{x^3} \cdot \sqrt[8]{x^3} \cdot \sqrt[16]{x^3} \ldots$ $\qquad D = R$

$y = x^1 \cdot x^{\frac{3}{2}} \cdot x^{\frac{3}{4}} \cdot x^{\frac{3}{8}} \cdot x^{\frac{3}{16}} \ldots$

$y = x^{1 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{16} + \ldots}$

$y = x^{1 + 3(\underbrace{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots}_{GŘ})} = x^{1+3s} = x^{1+3\cdot 1} = x^4$

$\left[\begin{array}{l} a_1 = \frac{1}{2} \quad q = \frac{1}{2} \quad |q|<1 \text{ pl.} \\ s = \frac{a_1}{1-q} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1 \end{array} \right]$

DÚ b) $y = 3\sqrt{3} \cdot \sqrt[4]{3} \cdot \sqrt[8]{3} \cdot \sqrt[16]{3} \ldots$

$y = 3 \cdot 3^{\frac{1}{2}} \cdot 3^{\frac{1}{4}} \cdot 3^{\frac{1}{8}} \cdot 3^{\frac{1}{16}} = 3^{1 + \underbrace{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots}_{GŘ}} = 3^{1+s} = 3^2 = 9$

$\left[\begin{array}{l} a_1 = \frac{1}{2} \quad q = \frac{1}{2} \quad |q|<1 \text{ pl.} \\ s = \frac{a_1}{1-q} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1 \end{array} \right]$

(13) Řešte v R

$x + 3x^2 + x^3 + 3x^4 + \ldots = \frac{5}{3}$ $\qquad D' = R$

rozdělíme na 2 řady

$x + x^3 + x^5 + \ldots$

$GŘ:\ a_1 = x \quad q = x^2 \quad |x^2| < 1$

$|x| < 1$

$x \in (-1,1)$ $\quad D'' (-1,1)$

$s_1 = \frac{x}{1-x^2}$

$D = D' \cap D''$

$D = (-1,1)$

$3x^2 + 3x^4 + 3x^6 + \ldots$

$3(\underbrace{x^2 + x^4 + x^6 + \ldots}_{GŘ})$

$GŘ:\ a_1 = x^2 \quad q = x^2 \quad |x^2| < 1$

$|x| < 1$

$x \in (-1,1)$

$s_2 = 3\frac{x^2}{1-x^2}$

rce má tvar: $s_1 + s_2 = \frac{5}{3}$

$\frac{x}{1-x^2} + \frac{3x^2}{1-x^2} = \frac{5}{3}$

$3x + 9x^2 = 5 - 5x^2$

$14x^2 + 3x - 5 = 0$

$x_{1/2} = \frac{-3 \pm \sqrt{9 + 4 \cdot 14 \cdot 5}}{28} = \frac{-3 \pm \sqrt{289}}{28}$

$= \frac{-3 \pm 17}{28} = \begin{cases} -\frac{20}{28} = -\frac{5}{7} \in D \\ \frac{14}{28} = \frac{1}{2} \in D \end{cases}$

$K = \{\frac{1}{2}, -\frac{5}{7}\}$